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# $\pi_*$-KERNELS OF LIE GROUPS

KEN-ICHI MARUYAMA

Abstract. We study a filtration on the group of homotopy classes of self maps of a compact Lie group associated with homotopy groups. We determine these filtrations of $SU(3)$ and $Sp(2)$ completely. We introduce two natural invariants $lz_p(X)$ and $sz_p(X)$ defined by the filtration, where $p$ is a prime number, and compute the invariants for simple Lie groups in the cases where Lie groups are $p$-regular or quasi $p$-regular. We apply our results to the groups of self homotopy equivalences.

## Introduction

Let $[X, Y]$ be the set of based homotopy classes of maps from a space $X$ to a space $Y$. We denote by $\mathcal{Z}^n(X, Y)$ the subset of $[X, Y]$ consisting of all homotopy classes which induce the trivial homomorphism on homotopy groups in dimensions less than or equal to $n$. We denote by $\mathcal{Z}^\infty(X, Y)$ the subset of $[X, Y]$ consisting of all homotopy classes which induce the trivial homomorphism on all homotopy groups. For short we write $\mathcal{Z}^n(X)$ and $\mathcal{Z}^\infty(X)$ if $X = Y$. In stable theory the set $\mathcal{Z}^\infty(\mathbf{X}, \mathbf{Y})$ has been previously considered by Christensen [2], where $\mathbf{X}, \mathbf{Y}$ are spectra. He calls the elements of $\mathcal{Z}^\infty(\mathbf{X}, \mathbf{Y})$ ghosts. Indeed there is a conjecture by Freyd [4] which states that $\mathcal{Z}^\infty(\mathbf{X}, \mathbf{Y})$ is trivial for finite spectra. On the other hand, in the unstable case the situation is quite different. $\mathcal{Z}^\infty(X, Y)$ is often nontrivial, even infinite for some spaces.

Let us consider the case where $Y$ is an $H$-space. If $Y$ is an $H$-space, $[X, Y]$ is an algebraic loop which is actually a group if $Y$ is homotopy associative by a result of James [5]. In this case, $\mathcal{Z}^n(X, Y)$ is a normal subgroup of $[X, Y]$ for any $n$. It is not commutative in general, but nilpotent in many cases. A main object of this paper is a study of the groups $\mathcal{Z}^n(X)$ for simple Lie groups. We will show that these groups have rather simple structure and can be computable in many cases. For the rank 2 case, we obtain the following theorem.

**Theorem 3.3.**

$$\mathcal{Z}^\infty(SU(3)) = \mathcal{Z}^n(SU(3)) \cong \mathbf{Z}_{12} \textit{ for } n \geq 5, \tag{1}$$

$$\mathcal{Z}^\infty(Sp(2)) = \mathcal{Z}^n(Sp(2)) \cong \mathbf{Z}_{120} \textit{ for } n \geq 7. \tag{2}$$

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When $G$ is a compact connected Lie group it is known [7] that $\mathcal{Z}^\infty(G)$ is equal to $\mathcal{Z}^n(G)$ for some $n$. The smallest such $n$ is written $sz(G)$ and called the stability of the descending series $\{\mathcal{Z}^n(G)\}$. We denote by $lz(G)$ the length of $\{\mathcal{Z}^n(G)\}$. We also define invariants of $G$ by $sz_p(G) = sz(G_p)$ and $lz_p(G) = lz(G_p)$ respectively, where $p$ is a prime number and $G_p$ is the $p$-localization of the space $G$. We will show that if $G$ is $p$-regular or quasi $p$-regular, then $sz_p(G)$ and $lz_p(G)$ are determined by its rational type:

**Theorem 5.1.** *Let $G$ be a compact connected, simply connected simple Lie group, $H^*(G;\mathbf{Q}) \cong \Lambda_{\mathbf{Q}}(x_1, ..., x_r)$ with $\deg x_i = n_i$, $n_1 \leq \cdots \leq n_r$. If $G$ is quasi $p$-regular, then $sz_p(G) = n_r$ for all $G$ and*

(1) *If $G$ is not isomorphic to $Spin(4n)$, then $lz_p(G) = r = \operatorname{rank} G$.*
(2) *If $G$ is isomorphic to $Spin(4n)$, then $lz_p(G) = r - 1 = \operatorname{rank} G - 1$.*

We note that $sz(G) \geq sz_p(G)$ and $lz(G) \geq lz_p(G)$ for an arbitrary prime number $p$ (see section 4). Our theorems prompt the following question.

**Question.** Let $G$ be a compact, connected simply connected simple Lie group such that $H^*(G;\mathbf{Q}) \cong \Lambda_{\mathbf{Q}}(x_1, ..., x_r)$ with $\deg x_i = n_i$, $n_1 \leq \cdots \leq n_r$. Is $sz(G) = n_r$?

We now briefly summarize the contents of this paper. In section 1 we collect some facts about homotopy sets of $H$-spaces. In section 2 we determine two numerical invariants $sz(X)$ and $lz(X)$ for some elementary spaces. In section 3 we consider $\mathcal{Z}^n(G)$ for rank 2 Lie groups $SU(3)$, $Sp(2)$ and $G_2$. We completely determine the group structures of $\mathcal{Z}^n(G)$ for $SU(3)$ and $Sp(2)$ for all $n$. In section 4 we give $p$-local consideration to $\mathcal{Z}^n(G)$ and $\mathcal{Z}^\infty(G)$ for simple Lie groups, and then we obtain $p$-local information on the invariants $sz(G)$ and $lz(G)$ when $G$ is $p$-regular. In section 5 a result in section 4 is generalized by using quasi $p$-regularity of Lie groups. In section 6 we give an application of the results in the previous sections. We will study the groups of self homotopy equivalences and their subgroups. The groups are closely related to the groups $\mathcal{Z}^n(G)$ when $G$ is a finite $H$-space. In particular, we will determine $\mathcal{E}^\infty_\#(G)$, the group of homotopy classes which induce the identity on all homotopy groups for $SU(3)$ and $Sp(2)$.

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## 1. Preliminaries

In this section we fix our notation and give some general results. Throughout this paper all spaces are connected and based. All maps and homotopies preserve base points. We do not distinguish notationally between a map and its homotopy class. For spaces $X$ and $Y$, we let $[X, Y]$ denote the pointed set of homotopy classes of maps from $X$ to $Y$. As we noted in our Introduction, if $Y$ is a homotopy associative $H$-space, then $[X, Y]$ is a group with the binary operation obtained from the $H$-structure of $Y$ [5]. We denote this operation "+". That is,

$$f + g = \mu_Y \circ (f \times g) \circ \Delta,$$

where $\mu_Y : Y \times Y \to Y$ is the $H$-structure of $Y$, $f, g \in [X, Y]$, and $\Delta : X \to X \times X$ is the diagonal map. Then for any map $h : W \to X$,

$$(f + g) \circ h = f \circ h + g \circ h \quad \text{in} \quad [W, Y]. \tag{1.1}$$

Let $\mathcal{Z}^n(X,Y) \subset [X,Y]$ denote the subset which consists of elements $f : X \to Y$ such that $f_* = 0 : \pi_i(X) \to \pi_i(Y)$ for $i \leq n$. We denote by $\mathcal{Z}^\infty(X,Y) \subset [X,Y]$ the subset which consists of elements $f : X \to Y$ such that $f_* = 0 : \pi_i(X) \to \pi_i(Y)$ for all $i$. We write $\mathcal{Z}^n(X)$ for $\mathcal{Z}^n(X,X)$ and $\mathcal{Z}^\infty(X)$ for $\mathcal{Z}^\infty(X,X)$. We call $\mathcal{Z}^n(X)$ *the nth $\pi_*$-kernel of $X$*.

**Proposition 1.1.** *Let $Y$ be a homotopy associative $H$-space. Then $\mathcal{Z}^n(X,Y)$ is a normal subgroup of the group $[X,Y]$.*

*Proof.* By (1.1), we obtain that $(f+g)_*(x) = f_*(x) + g_*(x)$ for $f, g \in [X,Y]$ and $x \in \pi_*(X)$. Thus the result follows. □

**Corollary 1.2.** *If $X$ is a connected finite homotopy associative $H$-space, then $\mathcal{Z}^n(X)$ is a nilpotent group.*

*Proof.* It is known that $[W,X]$ is a nilpotent group if $W$ is finite dimensional (cf. [18]). The result follows from Proposition 1.1. □

## 2. Numerical invariants

In this section we define two numerical invariants related to $\mathcal{Z}^*(X)$. We will investigate these invariants in later sections.

**Definition 2.1.** If there exists an integer $t \geq 1$ such that $\mathcal{Z}^n(X) = \mathcal{Z}^{n+1}(X)$ for $n \geq t$, the smallest such $t$ is written $sz(X)$. If no such integer exists we write $sz(X) = \infty$. We call $sz(X)$ the *stability* of $\mathcal{Z}^*(X)$.

**Definition 2.2.** The number of strict inclusions in $[X,X] \supset \mathcal{Z}^1(X) \supset \mathcal{Z}^2(X) \supset \cdots$ is denoted by $lz(X)$. We call $lz(X)$ the length of $\mathcal{Z}^*(X)$.

*Remark.* Clearly $sz(X) \geq lz(X)$. We may consider that $lz(X)$ reflects algebraic properties of $\mathcal{Z}^*(X)$ while $sz(X)$ is a geometric invariant.

We introduce two lemmas which will be needed in the sequel.

**Lemma 2.3.** *Let $\prod_{i=1}^r S_\ell^{n_i}$ be the product of spheres localized at $\ell$ with $n_1 \leq \cdots \leq n_r$. Then $sz(\prod_{i=1}^r S_\ell^{n_i}) = n_r$, and $lz(\prod_{i=1}^r S_\ell^{n_i}) = \#\{n_1, ..., n_r\}$ (= the number of distinct integers in $\{n_1, ..., n_r\}$), where $\ell$ is an arbitrary collection of prime numbers.*

*Proof.* The inclusion $\bigvee_{i=1}^r S_\ell^{n_i} \to \prod_{i=1}^r S_\ell^{n_i}$ induces an epimorphism on homotopy groups. It follows that $f \in \mathcal{Z}^{n_r}(\prod_{i=1}^r S_\ell^{n_i})$ implies $f\,|_{\bigvee_{i=1}^r S_\ell^{n_i}} \simeq *$ and thus $f \in \mathcal{Z}^\infty(\prod_{i=1}^r S_\ell^{n_i})$. Therefore we obtain the first equality. For the second equality, note that there exists a map $* \times id : \prod_{i=1}^j S_\ell^{n_i} \times \prod_{i=j+1}^r S_\ell^{n_i} \to \prod_{i=1}^j S_\ell^{n_i} \times \prod_{i=j+1}^r S_\ell^{n_i}$ which belongs to $\mathcal{Z}^{n_j}(\prod_{i=1}^r S_\ell^{n_i})$ but does not belong to $\mathcal{Z}^{n_{j+1}}(\prod_{i=1}^r S_\ell^{n_i})$ if $n_j \neq n_{j+1}$. Thus $lz(\prod_{i=1}^r S_\ell^{n_i}) \geq \#\{n_1, ..., n_r\}$. Now assume that $n_j \neq n_{j+1}$. Since $\pi_k(\bigvee_{i=1}^j S_\ell^{n_i}) \to \pi_k(\prod_{i=1}^r S_\ell^{n_i})$ is surjective for $k \leq n_{j+1} - 1$, we obtain

$$\mathcal{Z}^{n_j}(\prod_{i=1}^r S_\ell^{n_i}) = \cdots = \mathcal{Z}^{n_{j+1}-1}(\prod_{i=1}^r S_\ell^{n_i}).$$

Here we should note that if $f \in \mathcal{Z}^{n_j}(\prod_{i=1}^r S_\ell^{n_i})$, then the restriction $f$ to $\bigvee_{i=1}^j S_\ell^{n_i}$ is null homotopic. Therefore $lz(\prod_{i=1}^r S_\ell^{n_i}) \leq \#\{n_1, ..., n_r\}$, and the result follows. □

We obtain a variation of the above lemma as follows.

**Lemma 2.4.** *Let $\prod_{i=1}^{r} B_i$ be a product space such that $\mathcal{Z}^\infty(B_i, B_j) = [B_i, B_j]$, for $i \neq j$.*

(1) *Assume that $sz(B_i) = s_i$. Then $sz(\prod_{i=1}^{r} B_i) = \max\{s_i\}$.*
(2) *Assume that $lz(B_i) = l_i$ and $\mathcal{Z}^{i_k}(B_i) \neq \mathcal{Z}^{i_k+1}(B_i)$ for $k = 1, ..., l_i$. Then $lz(\prod_{i=1}^{r} B_i) = \#\{i_k\}$, $i = 1, ..., r$, $k = 1, ..., l_i$.*

*Proof.* The proof is similar to that of Lemma 2.3. □

## 3. Lie groups of rank 2

In this section we study $\mathcal{Z}^n(X)$ or $\mathcal{Z}^\infty(X)$ for $SU(3)$, $Sp(2)$ and $G_2$. The self homotopy sets $[X, X]$ have been determined by Mimura and Ōshima [10] when $X = SU(3), Sp(2)$ and $[G_2, G_2]$ has been determined by Ōshima up to group extension in [15]. They have obtained the following results.

**Theorem 3.1** ([10, 15])**.** *Relative to the standard multiplications,*

$$[SU(3), SU(3)] \cong \Psi(12, 1),$$
$$[Sp(2), Sp(2)] \cong \Psi(120, 12).$$

*The following sequence is exact:*

$$0 \to \pi_{14}(G_2) \xrightarrow{q_{14}^*} [G_2, G_2] \to \Psi(2, 1) \to 0. \tag{3.1}$$

*Here $\Psi(m, n)$ is the group with generators $x, y, z$ and relations*

$$xz = zx,\ yz = zy,\ z^m = 1,\ [x, y] = z^n.$$

By Theorem 3.1 and the results of [10], we obtain

**Theorem 3.2.**

$$\mathcal{Z}^n(SU(3)) \cong \begin{cases} [SU(3), SU(3)], & \text{for } n < 3, \\ \mathbf{Z} \oplus \mathbf{Z}_{12}, & \text{for } 3 \le n < 5, \\ \mathbf{Z}_{12} \quad \text{generated by } z, & \text{for } n = 5, \end{cases} \tag{1}$$

$$\mathcal{Z}^n(Sp(2)) \cong \begin{cases} [Sp(2), Sp(2)], & \text{for } n < 3, \\ \mathbf{Z} \oplus \mathbf{Z}_{120}, & \text{for } 3 \le n < 7, \\ \mathbf{Z}_{120} \quad \text{generated by } z, & \text{for } n = 7. \end{cases} \tag{2}$$

*Proof.* We have $\pi_5(SU(3)) \cong \mathbf{Z}$, $\pi_8(SU(3)) \cong \mathbf{Z}_{12}$ and $\pi_7(Sp(2)) \cong \mathbf{Z}$, $\pi_{10}(Sp(2)) \cong \mathbf{Z}_{120}$ (see [11]). By [10], $[SU(3), SU(3)]$ and $[Sp(2), Sp(2)]$ are generated by

$$x : X \xrightarrow{p} S^n \xrightarrow{a} X, \tag{3.2}$$

$$y = id : X \to X, \tag{3.3}$$

$$z : X \xrightarrow{q} S^{n+3} \xrightarrow{b} X, \tag{3.4}$$

where $n = 5$ for $X = SU(3)$ and $n = 7$ for $X = Sp(2)$, $p : X \to S^n$ is the bundle projection, $q : X \to S^{n+3}$ is the projection to the top cell, and $a$ and $b$ are generators of the homotopy groups. Let us consider the $SU(3)$ case. By [9], $\pi_5(SU(3))$ is generated by the element $[2\iota_5]$ such that $p_*([2\iota_5]) = 2\iota_5$, where $\iota_5$ is the identity class of $\pi_5(S^5)$. Thus $a$ in (3.2) is equal to $[2\iota_5]$. Since we have

$$x_*([2\iota_5]) = [2\iota_5] \circ p \circ [2\iota_5] = 2[2\iota_5],$$

$x$ induces a nontrivial homomorphism on $\pi_5(SU(3))$. Clearly $x$ induces the trivial homomorphisms on $\pi_3(SU(3))$ and $\pi_4(SU(3))$, and hence $x \in \mathcal{Z}^4(SU(3))$ and $x \notin \mathcal{Z}^5(SU(3))$. The map $z$ induces the zero homomorphism on homotopy groups $\pi_i(SU(3))$, $i \leq 8$, as $z$ factors through $S^8$. Thus $\mathcal{Z}^n(SU(3)) \supset \mathbf{Z} \oplus \mathbf{Z}_{12}$ for $n < 5$. The reverse containment is clear. Therefore, $\mathcal{Z}^n(SU(3))$ for $3 \leq n < 5$ are generated by $x$ and $z$, similarly $\mathcal{Z}^n(SU(3))$ for $5 \leq n \leq 8$ are generated by $z$. The group structures are obtained by Theorem 3.1.

The case where $X = Sp(2)$ is proved by the same methods. □

Next we consider $\mathcal{Z}^\infty(G)$ for $G = SU(3)$ and $Sp(2)$. We can completely determine these groups. Our result is as follows.

**Theorem 3.3.**

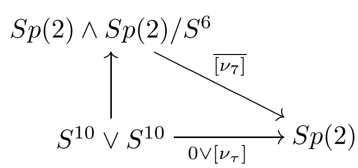
(1) $$\mathcal{Z}^\infty(SU(3)) = \mathcal{Z}^n(SU(3)) \cong \mathbf{Z}_{12} \textit{ for } n \geq 5,$$

(2) $$\mathcal{Z}^\infty(Sp(2)) = \mathcal{Z}^n(Sp(2)) \cong \mathbf{Z}_{120} \textit{ for } n \geq 7.$$

*Proof.* By Theorem 3.2, $\mathcal{Z}^5(SU(3))$ is generated by the commutator $[x, y]$. Since the group structure of $[SU(3), SU(3)]$ is induced from the multiplication of $SU(3)$, $z = [x, y]$ induces $z_* = x_* + y_* - x_* - y_* = 0$ on homotopy groups, and hence we obtain (1). Next we prove (2). Let $\omega_n$ denote the generator of the group $\pi_{n+3}(S^n) \cong \mathbf{Z}_{24}$, $n \geq 5$. According to [9], Proposition 4.1, $Sp(2) \wedge Sp(2)/S^6$ has the following homotopy type.

$$S^{10} \vee S^{10} \cup_\xi C(S^{13} \vee S^{19}) \vee S^{13} \cup_{2\omega_{13}} e^{17} \vee S^{13} \cup_{2\omega_{13}} e^{17},$$

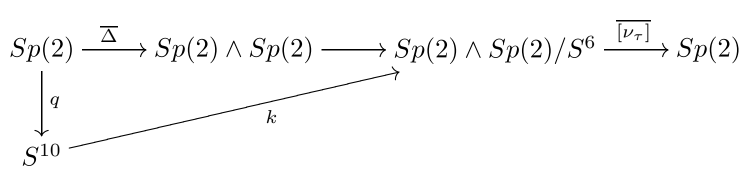
where $\xi = 2\omega_{10} + 4\omega_{10} + \delta$, and $\delta$ is a Whitehead product. From [11] we see that $[\nu_7] \circ 4\omega_{10} = 0$. Thus the generator $[\nu_7]$ of $\pi_{10}(Sp(2)) \cong \mathbf{Z}_{120}$ extends to $Sp(2) \wedge Sp(2)/S^6$. We denote by $\overline{[\nu_7]}$ an extension of $[\nu_7]$:




Now we consider the composition

(3.5) $$Sp(2) \xrightarrow{\bar{\Delta}} Sp(2) \wedge Sp(2) \to Sp(2) \wedge Sp(2)/S^6 \xrightarrow{\overline{[\nu_7]}} Sp(2),$$

where $\bar{\Delta}$ is the reduced diagonal map. By connectivity $\bar{\Delta}$ induces a map $k$ which makes the following diagram commutative:




The map $k$ is homotopic to $i_1 + i_2 : S^{10} \to Sp(2) \wedge Sp(2)/S^6$, where $i_1$ and $i_2$ are the inclusions. As $\overline{[\nu_7]}k = [\nu_7]$, the composition (3.5) is homotopic to the generator $z$ of $\mathcal{Z}^7(Sp(2)) \cong \mathbf{Z}_{120}$. Since $\bar{\Delta} \in \mathcal{Z}^\infty(Sp(2), Sp(2) \wedge Sp(2))$, we obtain $z \in \mathcal{Z}^\infty(Sp(2))$. □

**Corollary 3.4.** *$sz(SU(3)) = 5$, $lz(SU(3)) = 2$. $sz(Sp(2)) = 7$, $lz(Sp(2)) = 2$.*

We obtain a corresponding result to Theorem 3.2 for $G_2$ as follows.

**Theorem 3.5.**

$$\mathcal{Z}^n(G_2) \cong \begin{cases} [G_2, G_2], & \textit{for } n < 3, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_{21} \oplus \mathbf{Z}, & \textit{for } 3 \leq n < 11, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_{21}, & \textit{for } 11 \leq n \leq 14. \end{cases}$$

*Proof.* In the exact sequence (3.1) in Theorem 3.1, $\pi_{14}(G_2)$ is isomorphic to $\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{21}$, and the group $\Psi(2,1) \cong [G_2^{11}, G_2]$ is generated by the elements as follows [15]:

$$i_{11} : G_2^{11} \to G_2, \tag{3.6}$$

$$q_{11,6}^* \gamma' : G_2^{11} \xrightarrow{q_{11,6}} G_2^{11}/G_2^6 \to G_2, \tag{3.7}$$

$$q_{11}^* j_*[\nu_5^2] : G_2^{11} \xrightarrow{q_{11}} S^{11} \xrightarrow{j_*[\nu_5^2]} G_2, \tag{3.8}$$

where $j : SU(3) \to G_2$ is the inclusion map. The exact sequence (3.1) gives rise to the short exact sequence

$$0 \to \pi_{14}(G_2) \xrightarrow{q_{14}^*} \mathcal{Z}^n(G_2) \xrightarrow{i_{11}^*} \mathbf{Z} \oplus \mathbf{Z}_2 \to 0 \tag{3.9}$$

for $3 \leq n \leq 10$. Here the right term is generated by $q_{11,6}^*\gamma'$ and $q_{11}^* j_*[\nu_5^2]$ if $n \leq 7$, and it is generated by $q_{11}^* j_*[\nu_5^2]$ and $\epsilon q_{11,6}^*\gamma'$, $\epsilon = 1$ or $2$ if $7 < n \leq 10$. This can be seen as follows. First it is clear that the sequence is exact at $\pi_{14}(G_2)$. Since the inclusion map induces an epimorphism $\pi_i(G_2^{11}) \to \pi_i(G_2)$ for $i \leq 13$, any extensions of $q_{11,6}^*\gamma'$ and $q_{11}^* j_*[\nu_5^2]$ induce the trivial homomorphism on $\pi_i(G_2)$ for $i \leq 7$. Let $\alpha$ be an extension of $q_{11,6}^*\gamma'$, that is, $i_{11}^*\alpha = q_{11,6}^*\gamma'$. Then by [15], Theorem 2.2, $i_{11}^*[1,\alpha] = [i_{11}, q_{11,6}^*\gamma'] = q_{11}^* j_*[\nu_5^2]$. Thus $[1,\alpha]$ is an extension of $q_{11}^* j_*[\nu_5^2]$ and induces the trivial homomorphism on all homotopy groups as we saw in the proof of Theorem 3.3. Moreover, it is easy to see that $2\alpha$ induces the trivial homomorphism on $\pi_8(G_2)$ and $\pi_9(G_2)$. Thus $\mathcal{Z}^n(G_2)$, $n = 8, 9$, are generated by Im $q_{14}^*$ and $[1,\alpha]$ and $\epsilon\alpha$, where $\epsilon = 1$ if $\alpha_* = 0$ on $\pi_n(G_2)$ and $\epsilon = 2$ if $\alpha_* \neq 0$ on $\pi_n(G_2)$ $(n = 8, 9)$. Hence we obtain the exact sequence (3.9) (note that $\pi_{10}(G_2) = 0$). Now by [15],

$$[\alpha, [1, \alpha]] = 0.$$

Therefore, $\mathcal{Z}^n(G_2)$, $3 \leq n$, are abelian groups because the exact sequence (3.1) is central (see [15]) and so is (3.9). On the other hand,

$$G_2 \to G_2/G_2^9 \simeq S^{11} \vee S^{14} \to S^{11} \xrightarrow{j_*[\nu_5^2]} G_2$$

is also an extension of $q_{11}^* j_*[\nu_5^2]$ of order 2. Therefore, the above exact sequence (3.9) splits, since the groups $\mathcal{Z}^n(G_2)$ are abelian groups for $3 \leq n$. By [14], Lemma 5.8, $\bar{q}_{11,6}^*\gamma = 4\gamma'$, where $\gamma : S^{11} \to G_2$ is a generator of the direct summand $\mathbf{Z}$ of $\pi_{11}(G_2)$, and $\bar{q}_{11,6} : G_2^{11}/G_2^6 \to S^{11}$ is the projection map. Therefore an arbitrary extension of $q_{11,6}^*\gamma'$ to $\mathcal{Z}^n(G_2)$ induces a nontrivial homomorphism on $\pi_{11}(G_2)$. As the homotopy groups $\pi_n(G_2)$ are zero for $n = 12, 13$, $\mathcal{Z}^{11}(G_2) = \mathcal{Z}^{13}(G_2)$. Now (3.9) is restricted to the split exact aequence

$$0 \to \pi_{14}(G_2) \xrightarrow{q_{14}^*} \mathcal{Z}^{13}(G_2) \xrightarrow{i_{11}^*} \mathbf{Z}_2 \to 0.$$

This is because $\operatorname{Im} q_{14}^* \subset \mathcal{Z}^{13}(G_2)$ by definition and $[1,\alpha]$ induces the trivial homomorphism on all the homotopy groups. Thus $\mathcal{Z}^{13}(G_2)$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_{21}$ from the above argument. Similarly, $\operatorname{Im} q_{14}^* \subset \mathcal{Z}^{14}(G_2)$ and $[1,\alpha]_* = 0$, and so $\mathcal{Z}^{13}(G_2) = \mathcal{Z}^{14}(G_2)$. This completes the proof. □

## 4. $p$-local cases

For a set of prime numbers $P$ and a nilpotent space (or group) $X$, we denote by $X_P$ its $P$-localization. In particular, let $X_0$ denote the rationalization of a nilpotent space (or group) $X$. If $X$ is a finite $H$-space, its rational cohomology $H^*(X;\mathbf{Q})$ is isomorphic to $\Lambda(x_1, ..., x_\ell)$, the exterior algebra over $\mathbf{Q}$ with $\deg x_i$ odd. We call $\ell$ the rank of $X$ and ($\deg x_1$,...,$\deg x_\ell$) the type of $X$. It is known that $X_0$ is homotopy equivalent to $\prod_{i=1}^{\ell} S_0^{\deg x_i}$. Let us begin with definitions of the local stability and the local length of $\mathcal{Z}^*(X)$.

**Definition 4.1.** Let $X$ be a nilpotent space, and $P$ a set of prime numbers. We define two invariants of $X$ by $sz_P(X) = sz(X_P)$ and $lz_P(X) = lz(X_P)$.

**Lemma 4.2.** *For an $H$-space $X$ we define a map*

$$T : [X,X] \to [X,X] \tag{4.1}$$

*by $T(f) = 1_X + f$. If $X$ is a finite $H$-space, then $T$ induces a bijection $\mathcal{Z}^n(X_P) \to \mathcal{E}_\#^n(X_P)$ for $n \geq \dim X$, where $P$ is a set of prime numbers and $\mathcal{E}_\#^n(X_P)$ is the group of homotopy classes of self homotopy equivalences which induce the identity map on $\pi_i(X_P)$ for $i \leq n$.*

*Proof.* When $X$ is an $H$-space which is a CW complex, then $[X_P, X_P]$ is an algebraic loop by a result of [5], and $T$ is bijective on $[X_P, X_P]$. If $n \geq \dim X$, then $T(f)$ is a homotopy equivalence for $f \in \mathcal{Z}_\#^n(X_P)$, thus $T(f) \in \mathcal{E}_\#^n(X_P)$. □

We obtain the following.

**Lemma 4.3.** *Let $P$ be a set of prime numbers.*

1. *If $X$ is a finite $H$-space, then $sz_P(X)$ and $lz_P(X)$ are finite.*
2. *If $X$ is a finite homotopy associative $H$-space, then $\mathcal{Z}^n(X)_P \cong \mathcal{Z}^n(X_P)$ for any $n \geq 0$ and for $n = \infty$.*

*Proof.* By [7], the sequence $\{\mathcal{E}_\#^n(X_P)\}$ has finite length for a finite rational $H$-space $X$, so (1) follows from Lemma 4.2. (2) is obtained in [7]. □

*Remark.* We remark that if $X$ is a finite $H$-space, then $sz(X) = \max_p\{sz_p(X)\}$. Contrarily, $lz(X) \neq \max_p\{lz_p(X)\}$ in general. The following example is due to the referee:

$$\mathbf{Z}_{2\cdot 3\cdot 5\cdot 7} \supset \mathbf{Z}_{2\cdot 3\cdot 5} \supset \mathbf{Z}_{2\cdot 3} \supset \mathbf{Z}_2.$$

The sequence has length 3 even though at each prime its length is $\leq 1$.

We now completely determine the odd primary part of the group $\mathcal{Z}^*(G_2)$.

**Theorem 4.4.** *$\mathcal{Z}^\infty(G_2)_{(odd)} = \mathcal{Z}^n(G_2)_{(odd)} \cong \mathbf{Z}_{21}$ for $n \geq 11$.*

*Proof.* Since $\mathcal{Z}^{11}(G_2)_{(odd)} \cong \mathbf{Z}_{21}$, we only have to consider the 3 and 7 primary cases. First we consider the case when $p = 3$. It is known that $(G_2)_3$ is equivalent to the principal $S^3$-bundle over $S^{11}$ with the characteristic element $\alpha_2$. Thus $G_2$ at 3 has the following cell structure:

$$(G_2)_3 \simeq (S^3 \cup_{\alpha_2} e^{11} \cup_\rho S^{14})_3$$

By results of James [6], $\Sigma\rho$ is homotopic to $\Sigma i \circ J(\chi)$, where $i : S^3 \to G_2$, and $J(\chi)$ is the $J$-image of the characteristic element $\chi \in \pi_{10}(O(4))$ of the bundle. Therefore $J(\chi) = \alpha_1(4) \circ \alpha_2(7)$ or 0 by [17]. By using the fact that $\alpha_1(7) \circ \alpha_2(10) = 0$ [17], $\Sigma^4\rho$ is trivial. Consider the following homotopy commutative diagram:

(4.2)
$$\begin{array}{ccccc}
G_2 & \xrightarrow{(1\times p_{11})\Delta} & G_2 \times S^{11} & \xrightarrow{\wedge} & G_2 \wedge S^{11} \\
& \searrow^{((1\times p_{11})\Delta)'} & \uparrow & & \uparrow \\
& & (G_2 \times S^{11})^{14} & \xrightarrow{\wedge'} & S^3 \wedge S^{11}
\end{array}$$

where $\Delta$ is the diagonal map, $\wedge$ is the projection and $p_{11}$ is the bundle projection, and $((1 \times p_{11})\Delta)'$ and $\wedge'$ are maps induced from $(1 \times p_{11})\Delta$ and $\wedge$ by dimensional reasons. Since $\wedge' \circ ((1 \times p_{11})\Delta)'$ induces the degree one map on $H^{14}(G_2)$, the map is homotopic to $q$, the projection map to the top cell. As we saw that $\Sigma^4\rho = 0$, $(G_2 \wedge S^{11})_3 \simeq (S^{14} \cup_{\alpha_2} e^{22} \vee S^{25})_3$. From the results in the previous section we see that $\mathcal{Z}^{11}(G_2)_3 \cong \mathbf{Z}_3$ is generated by the composition

$$q^*(\alpha_3) : G_2 \to S^{14} \xrightarrow{\alpha_3} G_2.$$

Since $\pi_{21}(G_2) = 0$ [8], $\alpha_3$ extends to $S^{14} \cup_{\alpha_2} e^{22}$, namely, we obtain the following commutative diagram localized at 3:

$$\begin{array}{ccc}
S^{14} \cup_{\alpha_2} e^{22} & & \\
\uparrow & \searrow^{\overline{\alpha_3}} & \\
S^{14} & \xrightarrow[\alpha_3]{} & G_2 \\
{\scriptstyle \alpha_2}\uparrow & & \\
S^{21} & &
\end{array}$$

We have the following commutative diagram at 3:

(4.3)
$$\begin{array}{ccccccc}
G_2 & \xrightarrow{(1\times p_{11})\Delta} & G_2 \times S^{11} & \xrightarrow{\wedge} & G_2 \wedge S^{11} & & \\
& \searrow^{((1\times p_{11})\Delta)'} & \uparrow & & \uparrow & \searrow^{\overline{\alpha_3}\vee 0} & \\
& & (G_2 \times S^{11})^{14} & \xrightarrow{\wedge'} & S^3 \wedge S^{11} & \xrightarrow{\alpha^3} & G_2
\end{array}$$

In (4.3), the lower composition is homotopic to $q^*(\alpha_3)$, and the upper composition is an element of $\mathcal{Z}^\infty(G_2)$ since $\wedge$ induces the trivial map on homotopy groups. Thus we obtain the result for $p = 3$. For $p = 7$, $G_2$ is 7-equivalent to the product $S^3 \times S^7$, and hence it easily follows that $\mathcal{Z}^\infty(G_2)_7 \cong \mathcal{Z}^\infty((G_2)_7) \cong \mathcal{Z}^\infty((S^3 \times S^{11})_7) \cong \mathcal{Z}^\infty(S^3 \times S^{11})_7 \cong \mathcal{Z}^{11}(S^3 \times S^{11})_7 \cong \mathbf{Z}_7$ by Lemma 2.3 and Lemma 4.3. □

**Corollary 4.5.** $sz_{(odd)}(G_2) = 11$, $lz_{(odd)}(G_2) = 2$.

Next we will study $\mathcal{Z}^\infty(G)_p$, $sz_p(G)$ and $lz_p(G)$ for compact Lie groups. Using Lemma 4.3 we can obtain a finiteness property of $\mathcal{Z}^\infty(G)$ for compact Lie groups $G$.

**Proposition 4.6.** *Let $G$ be a compact connected, simply connected simple Lie group. Then $\mathcal{Z}^\infty(G)$ is finite if and only if $G$ is isomorphic to one of the following groups:*

(1) $SU(n)$ *for* $n < 8$.
(2) $Spin(n)$ *for* $n < 29$ *and* $n \neq 14,\ 18,\ 22,\ 26$.
(3) $Sp(n)$ *for* $n < 14$.
(4) *Exceptional Lie groups other than* $E_6$.

*Proof.* As $G_0$ is homotopy equivalent to $\prod_{i=1}^r S_0^{n_i}$, where $n_1 \le \cdots \le n_r$ and $n_i$ is an odd integer, $\mathcal{Z}^\infty(G)_0$ is isomorphic to $\mathcal{Z}^\infty(\prod_{i=1}^r S_0^{n_i})$. The latter group is isomorphic to the subgroup of $[\prod_{i=1}^r S_0^{n_i}, \prod_{i=1}^r S_0^{n_i}] \cong \bigoplus_{i=1}^r H^{n_i}(\prod_{i=1}^r S_0^{n_i})$ generated by decomposable elements of dimensions $\le n_r$. Our result is derived from the rational types and observation of the Poincaré polynomials of these groups. □

We have the following remark for nonsimply connected Lie groups.

*Remark.* (1) If $G$ is a compact connected, simple Lie group, then $\mathcal{Z}^\infty(G)$ is finite if and only if $\mathcal{Z}^\infty(\tilde{G})$ is finite, where $\tilde{G}$ is the universal cover of $G$. For example, $\mathcal{Z}^\infty(SO(n))$ is finite if and only if $\mathcal{Z}^\infty(Spin(n))$ is finite.

(2) We also easily see that $\mathcal{Z}^\infty(U(n))$ is finite if and only if $n < 5$

In the above proposition we used 0-regularity of Lie groups. Recall that a Lie group $G$ is called $p$-regular if and only if $G_p$ is homotopy equivalent to a product space $\prod_{i=1}^r S_p^{n_i}$, where $p$ is a prime and $n_i$ is an odd integer. We always assume that $n_1 \le \cdots \le n_r$. Next we apply $p$-regularity of Lie groups to obtain information on $sz(G)$ and $lz(G)$.

**Proposition 4.7.** *Let $G$ be a compact connected, simply connected simple Lie group. If $G$ is $p$-regular, then $sz_p(G) = sz_0(G)$, $lz_p(G) = lz_0(G)$. Thus $sz_p(G) = n_r$ for all $G$ and*

(1) *If $G$ is not isomorphic to $Spin(4n)$, then $lz_p(G) = r = \mathrm{rank}\ G$.*
(2) *If $G$ is isomorphic to $Spin(4n)$, then $lz_p(G) = r - 1 = \mathrm{rank}\ G - 1$.*

*Proof.* It is clear from Lemma 2.3 and Lemma 4.3 that $sz_p(G) = sz_0(G) = n_r$ and $lz_p(G) = lz_0(G)$ for any $p$-regular Lie group $G$. Let us consider the case $G = Spin(2n)$. If $Spin(2n)$ is $p$-regular, then

$$Spin(2n)_p \simeq \prod_{i=2}^{n} S_p^{4i-5} \times S_p^{2n-1}.$$

Therefore

$$lz_p(Spin(2n)) = \#\{3, 7, ..., 4n-5, 2n-1\} = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

The remaining cases are similar. □

*Remark.* (1) Since $SO(n)_p \simeq Spin(n)_p$ for an odd prime $p$, we obtain in this case

$$sz_p(SO(n)) = sz_p(Spin(n)),\ lz_p(SO(n)) = lz_p(Spin(n)).$$

(2) It is known that $U(n)$ is homeomorphic to $S^1 \times SU(n)$. From the cofibration

$$S^1 \vee SU(n) \to S^1 \times SU(n) \to S^1 \wedge SU(n),$$

there exists an exact sequence

$$0 \to [S^1 \wedge SU(n), SU(n)] \to \mathcal{Z}^i(S^1 \times SU(n)) \to \mathcal{Z}^i(SU(n)) \to 0$$

for $i \geq 1$. Thus $sz(U(n)) = sz(SU(n))$ and $lz(U(n)) = lz(SU(n)) + 1$ for $n > 1$. These also follow from Lemma 2.4.

## 5. Quasi $p$-regularity

In the previous section we studied the group $\mathcal{Z}^*(G)_p$ for the case where $G$ is $p$-regular. In this section we will generalize Proposition 4.7. We will show that $p$-regularity can be replaced by quasi $p$-regularity. Quasi regularity of Lie groups has been studied in [13] and [11]. Let $B_n(p)$ be the $S^{2n+1}$-bundle over $S^{2n+2p-1}$ determined by the generator $\alpha_1$ of $\pi_{2n+2p-2}(S^{2n+1}) \simeq \mathbf{Z}_p$. $B_n(p)$ has a cell structure of the form

$$S^{2n+1} \cup_{\alpha_1} e^{2n+1+2(p-1)} \cup e^{4n+2+2(p-1)}.$$

A Lie group $G$ is called quasi $p$-regular if and only if $G$ is homotopy equivalent to a product of spheres and $B_n(p)$'s at a prime $p$:

$$\prod S_p^{m_i} \times \prod B_{n_j}(p)_p \simeq G_p.$$

**Theorem 5.1.** *Let $G$ be a compact connected, simply connected simple Lie group, $H^*(G;\mathbf{Q}) \cong \Lambda_{\mathbf{Q}}(x_1, ..., x_r)$ with $\deg x_i = n_i$, $n_1 \leq \cdots \leq n_r$. If $G$ is quasi $p$-regular, then $sz_p(G) = n_r$ for all $G$ and*

1. *If $G$ is not isomorphic to $Spin(4n)$, then $lz_p(G) = r = \text{rank } G$.*
2. *If $G$ is isomorphic to $Spin(4n)$, then $lz_p(G) = r - 1 = \text{rank } G - 1$.*

In the remainder of this section all spaces are localized at a prime number $p$ which we are considering. For example, $B_n(p)$ means $B_n(p)_p$. We quote from [17]

**Proposition 5.2.** *Let $p$ be an odd prime. Then $\pi_{2n+1+2k(p-1)-2}(S^{2n+1}) \cong \mathbf{Z}_p$ for $1 \leq n < k$, and $k = 2, \ldots, p-1$, $\pi_{2n+1+2k(p-1)-1}(S^{2n+1}) \cong \mathbf{Z}_p$ for $k = 1, 2, ..., p-1$, $n \geq 1$, and $\pi_{2n+1+t}(S^{2n+1}) = 0$ otherwise for $t < 2p(p-1) - 2$.*

**Lemma 5.3.** *Let $p$ be an odd prime. If $[S^{2n+1} \cup_{\alpha_1} e^{2n+2p-1}, X] = 0$, then $q^* : [S^{4n+2p}, X] \to [B_n(p), X]$ is surjective.*

*Proof.* The result is clear from the following exact sequence:

$$[S^{4n+2p}, X] \xrightarrow{q^*} [B_n(p), X] \xrightarrow{j^*} [S^{2n+1} \cup_{\alpha_1} e^{2n+2p-1}, X]. \tag{5.1}$$

Here the sequence is induced by the cofibration. □

**Lemma 5.4.** *Let $p$ be an odd prime, $n$ a positive integer less than $p$, and $m$ a positive integer such that $n \neq m$ and $n - m + p - 1 \neq 0$. Let $q : B_n(p) \to S^{4n+2p}$ be the projection map to the top cell. Then*

$$q^* : [S^{4n+2p}, S^{2m+1}] \to [B_n(p), S^{2m+1}], \tag{1}$$

$$q^* : [S^{4n+2p}, B_m(p)] \to [B_n(p), B_m(p)], \tag{2}$$

*are surjective.*

*Proof.* We will show that $[S^{2p-1} \cup_{\alpha_1} e^{4p-3}, X] = 0$ and then apply Lemma 5.3, where $X$ is $S^{2m+1}$ or $B_m(p)$. First note that $2(n-m) < 2p-3$ and $2n+1 < 2m+2p-1$. It follows that

$$\pi_{2n+1}(S^{2m+1}) = 0,\ \pi_{2n+1}(S^{2m+2p-1}) = 0. \tag{5.2}$$

By using the fibration

$$S^{2m+1} \to B_m(p) \to S^{2m+2p-1} \tag{5.3}$$

we also have $\pi_{2n+1}(B_m(p)) = 0$, and hence we obtain the exact sequence

$$[S^{2n+2}, X] \xrightarrow{\alpha_1^*} [S^{2n+2p-1}, X] \to [S^{2n+1} \cup_{\alpha_1} e^{2n+2p-1}, X] \to 0 \tag{5.4}$$

for $X = S^{2m+1}$ or $B_n(p)$. As $2n-2m+2p-2 < 2p(p-1)-2$, we apply Proposition 5.2. If $\pi_{2n+2p-1}(S^{2m+1})$ is nontrivial, then there must be an integer $k > m$ such that

$$2n - 2m + 2p - 2 = 2k(p-1) - 2.$$

Thus $(n-m) > m(p-1)-p$, and hence $n > p(m-1)$. Since $p > n$, we have $m = 1$. Therefore, if $m > 1$, then the homotopy group $\pi_{2n+2p-1}(S^{2m+1})$ is trivial, moreover since $2(n-m) < 2p-3$, $\pi_{2n+2p-1}(S^{2m+2p-1})$ is trivial, and hence we see that $\pi_{2n+2p-1}(B_m(p))$ is trivial from the fibration (5.3). Thus $[S^{2n+1} \cup_{\alpha_1} e^{2n+2p-1}, X] = 0$ for $X = S^{2m+1}$ or $B_m(p)$ for $m > 1$ by (5.4). If $m = 1$, then $n-1 = k(p-1)-p$. Since $p > n$, we have $p-1 > k(p-1)-p$, so $k = 2$ and $n = p-1$. Since $\pi_{2p}(S^3)$ is isomorphic to $\mathbf{Z}_p$ generated by $\alpha_1$, and $\pi_{4p-3}(S^3)$ is isomorphic to $\mathbf{Z}_p$ generated by $\alpha_1^2$ [17], the homomorphism

$$\alpha_1^* : [S^{2p}, S^3] \to [S^{4p-3}, S^3]$$

is an isomorphism, thus we obtain that $[S^{2p-1} \cup_{\alpha_1} e^{4p-3}, S^3] = 0$ by (5.4). Finally, $\pi_{4p-3}(B_1(p))$ is trivial by the fibration (see [13])

$$S^3 \to B_1(p) \to S^{2p+1}.$$

Hence $[S^{2p-1} \cup_{\alpha_1} e^{4p-3}, B_1(p)] = 0$. Now Lemma 5.4 follows from Lemma 5.3. □

The following result follows from the results of [6].

**Lemma 5.5** ([12])**.** $\Sigma^2 B_n(p) \simeq S^{2n+3} \cup_{\alpha_1} e^{2n+2p+1} \vee S^{4n+2p+2}$.

By using Lemma 5.5 we obtain the next lemma which will play an essential role in this section.

**Lemma 5.6.** *If* $\pi_{4n+4p-3}(X) = 0$*, then* $\operatorname{Im} q^* \subset \mathcal{Z}^\infty(B_n(p), X)$ *for a space* $X$*, where* $q^* : [S^{4n+2p}, X] \to [B_n(p), X]$*.*

*Proof.* Our argument is analogous to that of the proof of Theorem 4.4. There exists a commutative diagram similar to (4.2)

$$\begin{array}{ccccc} B_n(0) & \xrightarrow{(1\times p_n)\Delta} & B_n(p) \times S^{2n+2p-1} & \xrightarrow{\wedge} & B_n(p) \wedge S^{2n+2p-1} \\ & \searrow^{((1\times p_n)\Delta)'} & \uparrow & & \uparrow \\ & & (b_n(p) \times S^{2n+2p-1})^{4n+2p} & \xrightarrow{\wedge'} & S^{2n+1} \wedge S^{2n+2p-1} \end{array} \tag{5.5}$$

where $p_n : B_n(p) \to S^{2n+2p-1}$ is the bundle projection, and $((1 \times p_n)\Delta)'$ is a map induced by $(1 \times p_n)\Delta$. If $\pi_{4n+4p-3}(X) = 0$, any map $a : S^{4n+2p} \to X$ extends to $B_n(p) \wedge S^{2n+2p-1} \simeq S^{4n+2p} \cup_{\alpha_1} e^{4n+4p-2} \vee S^{6n+4p-1}$. The lower map

$\wedge'((1\times p_n)\Delta)' : B_p(n) \to S^{4n+2p}$ is homotopic to $\epsilon\cdot q : B_n(p) \to S^{4n+2p}$ by cohomology reasons, where $\epsilon$ is an integer prime to $p$. We obtain the commutative diagram

$$
\begin{array}{ccccc}
(5.6) \quad B_n(p) & \xrightarrow{\wedge(1\times p_n)\Delta} & S^{4n+2p}\cup_{\alpha_1} e^{4n+4p-2}\wedge S^{6n+4p-1} & & \\
 & \searrow^{\wedge'((1\times p_n)\Delta)'} & \uparrow & \searrow^{\bar{a}\vee 0} & \\
 & & S^{2n+1}\wedge S^{2n+2p-1} & \xrightarrow{a} & X
\end{array}
$$

where $\bar{a}$ is an extension of $a$. The upper composition in (5.6) induces the zero on homotopy groups as $\wedge$ does. Therefore the lower composition in (5.6) induces the trivial map on homotopy groups, and so $q^*(a)\in\mathcal{Z}^\infty(B_n(p),X)$. □

**Lemma 5.7.** *Let $p$ be an odd prime, and $n$ and $m$ be positive integers with $n\neq m$.*

(1) $sz_p(B_n(p)) = 2n+2p-1$ *and* $lz_p(B_n(p)) = 2$ *for* $n\leq 2p-3,\ p\geq 5$, *and for* $n\leq 2, p=3$.

(2) $\mathcal{Z}^\infty(B_n(p),B_m(p)) = [B_n(p),B_m(p)]$ *for* $m,n<p,\ p\geq 3$.

(3) $\mathcal{Z}^\infty(B_n(p),S^{2m+1}) = [B_n(p),S^{2m+1}]$ *for* $n<p, n-m+p-1\neq 0, m\geq 3$, $p\geq 5$.

(4) $\mathcal{Z}^\infty(B_1(p),S^5) = [B_1(p),S^5]$ *for* $p\geq 3$.

*Proof.* (1) We consider the diagram

$$
\begin{array}{ccccc}
(5.7) & & & & [S^{2n+1},B_n(p)] \\
 & & & & \uparrow \\
[S^{4n+2p},B_n(p)] & \xrightarrow{q^*} & [B_n(p),B_n(p)] & \xrightarrow{j^*} & [S^{2n+1}\cup_{\alpha_1} e^{2n+2p-1},B_n(p)] \\
 & & & & \uparrow \\
 & & & & [S^{2n+2p-1},B_n(p)]
\end{array}
$$

The groups $[S^{2n+1},B_n(p)]$ and $[S^{2n+2p-1},B_n(p)]$ are both isomorphic to $\mathbf{Z}_{(p)}$, the integers localized at $p$. The group $[S^{2n+2}\cup_{\alpha_1} e^{2n+2p},B_n(p)]$ is trivial, and hence $q^*$ is injective. Since $\pi_{4n+4p-3}(B_n(p))$ is trivial by Proposition 5.2 for $n\leq 2p-3$ and $p\geq 5$, $\operatorname{Im} q^*\in\mathcal{Z}^\infty(B_n(p))$ by Lemma 5.6. The groups $\pi_k(B_n(p))$ are trivial for $2n+1<k<2n+2p-3$ and the homotopy group $\pi_{2n+2p-2}(B_n(p))$ is isomorphic to $\mathbf{Z}_p$ which is generated by $i_*\alpha_1$, where $i : S^{2n+1}\to B_n(p)$ is the inclusion map and $i_* : \pi_{2n+2p-2}(S^{2n+1})\to\pi_{2n+2p-2}(B_n(p))$. Therefore $f\in\mathcal{Z}^{2n+1}(B_n(p))$ implies $f\in\mathcal{Z}^{2n+2p-2}(B_n(p))$. Thus $\mathcal{Z}^{2n+1}(B_n(p)) = \cdots = \mathcal{Z}^{2n+2p-2}(B_n(p))$, and we obtain the exact sequence

$$0\to[S^{4n+2p},B_n(p)]\xrightarrow{q^*}\mathcal{Z}^k(B_n(p))\to\mathbf{Z}_{(p)}\to 0$$

for $2n<k<2n+2p-1$. Finally we have an isomorphism

$$[S^{4n+2p},B_n(p)]\xrightarrow{q^*}\mathcal{Z}^{2n+2p-1}(B_n(p)),$$

and thus $\mathcal{Z}^{2n+2p-1}(B_n(p)) = \mathcal{Z}^\infty(B_n(p))$ by Lemma 5.6. By the above argument $lz_p(B_n(p)) = 2$. Since $\pi_{13}(B_1(3)) = 0$ and $\pi_{17}(B_2(3)) = 0$, the $p=3$ case is similar.

(2) According to Oka [13], $\pi_{4n+4p-3}(B_m(p)) = 0$ for $m, n < p$ and $p \geq 3$. If $n \neq m$, we obtain (2) from Lemma 5.4 and Lemma 5.6.

(3) As $4n + 4p - 2m - 4 < 2p(p-1) - 2$ for $m \geq 3$, we apply Proposition 5.2 and obtain (3) by Lemma 5.4 and Lemma 5.6.

(4) also follows from Proposition 5.2, Lemma 5.4 and Lemma 5.6. □

Now we will prove the main theorem in this section.

*Proof of Theorem* 5.1. By [12], $G$ is simultaneously quasi $p$-regular and not $p$-regular if and only if

$$\begin{array}{ll} 2n > p > n & \text{for } Sp(n), \\ n > p > \dfrac{n}{2} & \text{for } SU(n), \\ n-1 > p > \dfrac{n-1}{2} & \text{for } Spin(n), \\ 7 > p \geq 5 & \text{for } G_2, \\ 13 > p \geq 5 & \text{for } F_4,\ E_6, \\ 19 > p \geq 11 & \text{for } E_7, \\ 31 > p \geq 11 & \text{for } E_8. \end{array}$$

The assertion for classical groups $SU(n)$, $Sp(n)$ follows from Lemma 5.7, Lemma 2.4 and the following homotopy equivalences [13]:

$$\prod_{k=1}^{n} B_k(p)_p \times \prod_{k=n+1}^{p-1} S_p^{2k+1} \cong SU(n+p)_p \quad \text{for } n < p,$$

$$\prod_{k=1}^{n} B_{2k-1}(p)_p \times \prod_{k=n+1}^{(p-1)/2} S_p^{4k-1} \cong Sp(n + (p-1)/2)_p \text{ for } n \leq (p-1)/2.$$

We should note that we can apply Lemma 2.4 since $[S^{m_i}, B_{n_j}]$ are trivial for these cases. For the $Spin(n)$ case, we use the following equivalences for an odd prime $p$ (see [12]):

$$Spin(2n+1)_p \cong Sp(n)_p,$$
$$Spin(2n)_p \cong Spin(2n-1)_p \times S_p^{2n-1}.$$

Moreover, our assertion for the exceptional Lie groups follows from Lemma 5.7, Lemma 2.4 and by the results of [12] except for the cases $G = F_4, p = 5$, $G = E_6, p = 5$, $G = E_8, p = 11, p = 13$. For these cases there exist equivalences as follows [11]:

$$(F_4)_5 \cong (B_1(5) \times B_7(5))_5,$$
$$(E_6)_5 \cong (B_1(5) \times B_4(5) \times B_7(5))_5,$$
$$(E_8)_{11} \cong (B_1(11) \times B_7(11) \times B_{13}(11) \times B_{19}(11))_{11},$$
$$(E_8)_{13} \cong (B_1(13) \times B_7(13) \times B_{11}(13) \times B_{17}(13))_{13}.$$

We easily see that all sets $[B_n(5), B_m(5)], n \neq m,\ n, m \in \{1, 4, 7\}$, are trivial except for the case where $n = 4, m = 1$. For example, we consider $[B_1(5), B_7(5)]$.

We obtain the diagram which is analogous to (5.7)

$$\begin{array}{ccccc} & & & & [S^3, B_7(5)] \\ & & & & \uparrow \\ [S^{14}, B_7(5)] & \xrightarrow{q^*} & [B_1(5), B_7(5)] & \xrightarrow{j^*} & [S^3 \cup_{\alpha_1} e^{11}, B_7(5)] \\ & & & & \uparrow \\ & & & & [S^{11}, B_7(5)] \end{array}$$

The groups $\pi_{14}(B_7(5))$, $\pi_{11}(B_7(5))$ and $\pi_3(B_7(5))$ are trivial using Proposition 5.2 and the fibration

$$S^{15} \to B_7(5) \to S^{23},$$

and hence $[B_1(5), B_7(5)] = 0$. The other cases are similar. Thus $\mathcal{Z}^\infty(F_4)_5 = \mathcal{Z}^{23}(F_4)_5$ by Lemma 5.7(1) and Lemma 2.4. On the other hand, by the same argument as above, $[B_4(5), B_1(5)]$ is equal to $q^*\pi_{26}(B_1(5))$, which is nontrivial ($\cong \mathbf{Z}_5$). According to [13], $\pi_{33}(B_1(5)) = 0$ and we apply Lemma 5.6 and obtain $\mathcal{Z}^\infty(B_4(5), B_1(5)) = [B_4(5), B_1(5)]$. By Lemma 5.7(1) and the last equality, we obtain $\mathcal{Z}^\infty(E_6)_5 = \mathcal{Z}^{23}(E_6)_5$.

We have $[B_n(11), B_m(11)] = 0$ for $n, m$ such that $n \neq m$ and $n, m \in \{1, 7, 13, 19\}$ except for $[B_{13}(11), B_7(11)]$, and we also have $[B_n(13), B_m(13)] = 0$ for $n, m$ such that $n \neq m$ and $n, m \in \{1, 7, 11, 17\}$ except for $[B_{17}(13), B_{11}(13)]$. The sets $[B_{13}(11), B_7(11)]$ and $[B_{17}(13), B_{11}(13)]$ are both $q^*$-images. Since $\pi_{93}(B_7(11)) = 0 = \pi_{117}(B_{11}(13))$, $\mathcal{Z}^\infty(B_{13}(11), B_7(11)) = [B_{13}(11), B_7(11)]$ and $\mathcal{Z}^\infty(B_{17}(13), B_{11}(13)) = [B_{17}(13), B_{11}(13)]$ by Lemma 5.6, and hence we obtain the $E_8$ case. For the assertion for $lz_p(G)$, we should note that in any quasi $p$-regular type of our $G$,

$$G_p \simeq \prod S_p^{m_i} \times \prod B_{n_j}(p)_p,$$

all homotopy sets $[S^{m_i}, B_{n_j}]$ are trivial for dimensional reasons. Therefore we obtain the result for $lz_p(G)$ by Lemma 2.4 and Lemma 5.7(1). □

As noted in the above proof, we know when our Lie groups are quasi $p$-regular by the results of [12]. Our final remark in this section is as follows.

*Remark.* By Theorem 5.1 and Corollary 4.5, we obtain the following:

If $p > \frac{n}{2}$, $sz_p(SU(n)) = 2n-1$ and $lz_p(SU(n)) = n-1$.
If $p > n$, $sz_p(Sp(n)) = 4n-1$ and $lz_p(Sp(n)) = n$.
If $p > n-1$, $sz_p(Spin(2n-1)) = 4n-5$ and $lz_p(Spin(2n-1)) = n-1$.
If $p > n-1$ and $n$ is odd, $sz_p(Spin(2n)) = 4n-5$ and $lz_p(Spin(2n)) = n$.
If $p > n-1$ and $n$ is even, $sz_p(Spin(2n)) = 4n-5$ and $lz_p(Spin(2n)) = n-1$.
If $p > 2$, $sz_p(G_2) = 11$ and $lz_p(G_2) = 2$.
If $p > 3$, $sz_p(F_4) = 23, sz_p(E_6) = 23$ and $lz_p(F_4) = 4, lz_p(E_6) = 6$.
If $p > 7$, $sz_p(E_7) = 35$ and $lz_p(E_7) = 7$.
If $p > 7$, $sz_p(E_8) = 59$ and $lz_p(E_8) = 8$.

## 6. Application

In this section, we will study the groups of self homotopy equivalences of Lie groups and their subgroups associated to homotopy groups. We will apply the results in the previous sections to determine these groups.

**Lemma 6.1.** *If $X$ is a finite $H$-space such that $H^*(X)$ has no torsion and $H^*(X;\mathbf{Q}) = \Lambda_{\mathbf{Q}}(x_1,...,x_r)$, with $\deg x_i = n_i$, $n_1 \leq \cdots \leq n_r$, then $T$ induces a bijection $\mathcal{Z}^n(X_P) \to \mathcal{E}^n_\#(X_P)$ for $n \geq n_r$, where $T$ is the map defined in* (4.1) *and $P$ is a set of prime numbers.*

*Proof.* As was mentioned in Lemma 4.2, $T : [X_P, X_P] \to [X_P, X_P]$ is bijective. If $X$ is a finite $H$-space which satisfies our conditions, then we can assume that $H^*(X;\mathbf{Z}) = \Lambda_{\mathbf{Z}}(x_1,...,x_r)$, and hence $H^*(X_P;\mathbf{Z}_P) = \Lambda_{\mathbf{Z}_P}(x_1,...,x_r)$. Therefore $T(f)$ is a homotopy equivalence for $f \in \mathcal{Z}^n_\#(X_P)$ if $n \geq n_r$. □

**Theorem 6.2.** $\mathcal{E}^\infty_\#(SU(3)) = \mathcal{E}^n_\#(SU(3)) \cong \mathbf{Z}_{12}$ *for* $n \geq 5$, $\mathcal{E}^\infty_\#(Sp(2)) = \mathcal{E}^n_\#(Sp(2)) \cong \mathbf{Z}_{120}$ *for* $n \geq 7$.

*Proof.* By Theorem 3.3 and Lemma 6.1 $\mathcal{E}^n_\#(SU(3)) = \mathcal{E}^{n+1}_\#(SU(3))$ for $n \geq 5$ and $\mathcal{E}^n_\#(Sp(2)) = \mathcal{E}^{n+1}_\#(Sp(2))$ for $n \geq 7$. The group structures are obtained by the results of [16] (or [10]). □

More generally we obtain the following theorem which corresponds to Theorem 5.1. We note that $\mathcal{E}^n_\#(X)$, $n \geq \dim X$, is a nilpotent group for a finite nilpotent space by [3].

**Theorem 6.3.** *Let $G$ be a compact connected, simply connected simple Lie group, $H^*(G;\mathbf{Q}) \cong \Lambda_{\mathbf{Q}}(x_1,...,x_r)$ with $\deg x_i = n_i$, $n_1 \leq \cdots \leq n_r$. Assume that $G$ is quasi $p$-regular.*

(1) $\mathcal{E}^\infty_\#(G)_p = \mathcal{E}^{\dim G}_\#(G)_p$.

(2) *Moreover, if $G = SU(n)$ or $Sp(n)$, then $\mathcal{E}^{n_r}_\#(G)$ is a nilpotent group and* $\mathcal{E}^\infty_\#(G)_p = \mathcal{E}^{\dim G}_\#(G)_p = \mathcal{E}^{n_r}_\#(G)_p$.

*Proof.* (1) If $n \geq \dim G$, then $\mathcal{E}^n_\#(G)_p \cong \mathcal{E}^n_\#(G_p)$; see [7]. Thus the result follows from Lemma 4.2, Lemma 4.3 and Theorem 5.1.

(2) If $G = SU(n)$ or $Sp(n)$, then $H_*(G)$ is torsion free. Therefore $\mathcal{E}^{n_r}_\#(G)$ acts on homology nilpotently, hence by a result of [3] it is a nilpotent group. If $\mathcal{E}^{n_r}_\#(G)_p \cong \mathcal{E}^{n_r}_\#(G_p)$, we obtain

$$\mathcal{E}^\infty_\#(G)_p = \mathcal{E}^\infty_\#(G_p) = \mathcal{E}^{n_r}_\#(G_p) = \mathcal{E}^{n_r}_\#(G)_p,$$

where the second equality follows from Theorem 5.1 and Lemma 6.1. Thus it suffices to show the following assertion.

**Assertion.** The natural map $\mathcal{E}^{n_r}_\#(G) \to \mathcal{E}^{n_r}_\#(G_p)$ is the localization for the above prime $p$.

In the remaining we will prove this assertion. Consider the following commutative diagram:

(6.1)
$$\begin{array}{ccccc}
\mathcal{E}^{n_r}_\#(G) & \longrightarrow & \mathcal{E}^{n_r}_\#(G_p) & & \\
\uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle j_*\cong} & \nwarrow{\scriptstyle \cong} & \\
\mathcal{E}^\infty_\#(G) & \longrightarrow & \mathcal{E}^\infty_\#(G_p) & \xrightarrow{\cong} & \mathcal{E}^{\dim G}_\#(G_p)
\end{array}$$

As we saw in the proof of (1), $j_* : \mathcal{E}_\#^\infty(G_p) \to \mathcal{E}_\#^{n_r}(G_p)$ is an isomorphism, and $\mathcal{E}_\#^\infty(G) \to \mathcal{E}_\#^\infty(G_p)$ is the localization. Thus by (6.1), $\mathcal{E}_\#^{n_r}(G)_p \to \mathcal{E}_\#^{n_r}(G_p)$ is an epimorphism. We also have the following commutative diagram:

(6.2)
$$\begin{array}{ccc} \mathcal{Z}^{n_r}(G_p) & \xrightarrow{T} & \mathcal{E}_\#^{n_r}(G_p) \\ \uparrow & & \uparrow \\ \mathcal{Z}^{n_r}(G) & \xrightarrow{T} & \mathcal{E}_\#^{n_r}(G) \end{array}$$

where the vertical maps are homomorphisms induced by $p$-localization, and the left map is the localization at $p$ by Lemma 4.3(2). Let $q$ be a prime number and let $S_q$ be the $q$-Sylow subgroup of the subgroup of all the torsion elements of $\mathcal{Z}^{n_r}(G)$. We have the equalities

$$T(x) \circ T(y) = (1_G + y + x \circ (1_G + y)),$$
$$T((-x) \circ (1_G + x)^{-1}) \circ T(x) = 1_G,$$

for any $x, y \in S_q$. Thus $T(S_q)$ is a subgroup of $\mathcal{E}_\#^{n_r}(G)$ of order $|S_q|$ since $T$ is bijective. Moreover if $q_1$ and $q_2$ are different primes, then $T(S_{q_1} \times S_{q_2})$ is a subgroup of order $|S_{q_1} \times S_{q_2}|$. Thus, if $x$ is a torsion element of order $|x|$ and $|x|$ is prime to $p$, then the order of $T(x)$ is prime to $p$. It follows that the kernel of the homomorphism $\mathcal{E}_\#^{n_r}(G) \to \mathcal{E}_\#^{n_r}(G_p)$ is a finite subgroup whose order is prime to $p$, so the map is a localization map at $p$. □

*Remark.* Arkowitz and Strom [1] computed $\mathcal{E}_\#^\infty(G)_p$ when $G$ is $p$-regular in many cases.

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Department of Mathematics, Faculty of Education, Chiba University, Yayoicho, Chiba, Japan
*E-mail address*: maruyama@faculty.chiba-u.jp